A Sraffian Supermultiplier Model

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1 Overview

The Sraffian supermultiplier model was proposed by Serrano (1995) to integrate a Sraffian long-run equilibrium into a post-Keynesian growth model.[1] The model requires the long-run rate of capacity utilisation to settle on an exogenously given normal rate. This requires investment to fully adjust to any changes in economic activity so as to bring back actual utilisation to the desired normal rate. As a result, investment expenditures (in the long-run) are assumed to be free of any idiosyncratic components such as Keynesian 'animal spirits'. Long-run growth is then driven by those components of autonomous demand that do not create productive capacity – autonomous consumption in the simplest version of the model. An increase in the growth rate of autonomous consumption will stimulate economic activity and induce firms to adjust their expectations about long-run growth towards the new rate given by autonomous demand growth.

Income distribution is exogenous in this model. An increase in the wage share has an expansionary effect on economic activity and growth in the short-run as it increases consumption (investment is assumed to be independent of income distribution). However, this expansionary effect is only temporary as economic activity will eventually settle back on the normal rate of capacity utilisation, and the growth rate towards the rate given by autonomous demand growth. The absence of long-run effects of income distribution on output and growth constitutes a key difference between the Sraffian supermultiplier model and the post-Kaleckian model, in which there is no normal rate of capacity utilisation and no autonomous (non-capacity creating) demand.

This is a model of long-run steady state growth. In the steady state, all endogenous variables grow at the same rate.[2] The model contains two state variables that determine the model's

[1] See Blecker & Setterfield (2019, chap. 7), Dutt (2018), and Lavoie (2022, chap. 6) for introductions and overviews. Note that contrary to what the name may suggest, this is a one-sector model.

[2] All variables are normalised by the capital stock and thus rendered stationary.

dynamics: the ratio of autonomous demand to the capital stock (which changes during adjustment periods where the growth rate has not yet settled on the rate given by autonomous demand growth) and the expected growth rate of the capital stock, which sluggishly adjusts to the rate given by autonomous demand growth. We consider a continuous-time version of the model presented in Lavoie (2022, chap. 6.5.8).[3]

2 The Model

$$r_t = \pi u_t \tag{1}$$

$$s_t = -z_t + s_r r_t, \qquad s_r > 0 \tag{2}$$

$$c_t = u_t - s_t \tag{3}$$

$$g_t = g_t^0 + g_1(u_t - u_n), \qquad g_1 > 0 \tag{4}$$

$$u_t = c_t + g_t \tag{5}$$

$$\dot{g}_t^0 = \mu(g_t - g_t^0), \qquad \mu > 0 \tag{6}$$

$$\dot{z}_t = z_t(g_z - g_t), \tag{7}$$

where r, s, c, g, u, g^0, and z are the profit rate, the saving rate, the consumption rate, the investment rate, the rate of capacity utilisation, the expected growth rate, and the rate of autonomous demand, respectively. A dot over a variable represents the derivative with respect to time ($\dot{x} = \frac{dx}{dt}$).

Equation (1) decomposes the profit rate (total profits over capital stock) into the product of the profit share π (total profits over total output) and the rate of capacity utilisation (actual output over capital stock).[4] Note that the wage share is given by $1-\pi$. By equation (2), the economy-wide saving rate is given by the negative of the rate of autonomous demand (z), which in this version of the model is autonomous consumption, i.e. dissaving, and saving out of profits ($s_r r$). It is assumed that workers do not save. Equation (3) simply states that consumption is income not saved. According to equation (3), investment is determined by an autonomous component g_0 that will be specified below and by the deviation of capacity utilisation from its normal rate u_n. In other words, firms expand capacity whenever the

[3] Appendix A explains how continuous time models can be solved numerically.

[4] For simplicity, it is assumed that the capital-potential output ratio is equal to unity. This implies that the ratio of actual output to potential output is equal to the ratio of actual output to the capital stock, so that the latter can be taken as a measure of the rate of capacity utilisation.

actual rate of utilisation exceeds the desired normal rate. Equation (5) is the goods market equilibrium condition assuming that the rate of capacity utilisation adjusts to clear the goods market in the short run. Equation (6) is a key equation in the Sraffian supermultiplier approach, which posits that firms (sluggishly) adjust the expected growth rate to the actual growth rate. Finally, equation (7) is an identity that traces changes in the rate of autonomous demand that stem from (temporary) mismatches between the exogenously given growth rate of autonomous demand (g_z) and the actual growth rate.

3 Simulation

Table 1 reports the parameterisation used in the simulation. Besides a baseline (labelled as scenario 1), three further scenarios will be considered. In scenario 2, the growth rate of autonomous demand g_z increases. In scenario 3, the profit share π rises. In scenario 4, the normal rate of capacity utilisation u_n increases. The model is initialised at the equilibrium of the baseline parameterisation and the various shifts then occur in period 50.

Table 1: Parameterisation

Scenario	π	s_r	g_1	u_n	μ	g_z
1: baseline	0.35	0.8	0.2	0.75	0.08	0.02
2: rise in autonomous demand growth (g_z)	0.35	0.8	0.2	0.75	0.08	0.03
3: rise in profit share (π)	0.4	0.8	0.2	0.75	0.08	0.02
4: rise in normal rate of capacity utilisation (u_n)	0.35	0.8	0.2	0.8	0.08	0.02

Figures 1-3 depict the response of the three main endogenous variables to changes in the exogenous variables. In the second scenario (solid line), the growth rate of autonomous demand increases from 2% to 3%. As a result, the rate of capacity temporarily increases but then returns to the level given by the normal rate, as the rate of autonomous demand falls due to the increase in the capital stock. By contrast, the growth rate permanently settles to the new rate given by the autonomous rate.

In the third scenario (dashed line), the profit share rises, which initially has a contractionary effect on the rate of utilisation and growth. Both variables then briefly overshoot due to the increase in the autonomous demand rate and then return to their previous values.

Figure 1: Rate of capacity utilisation

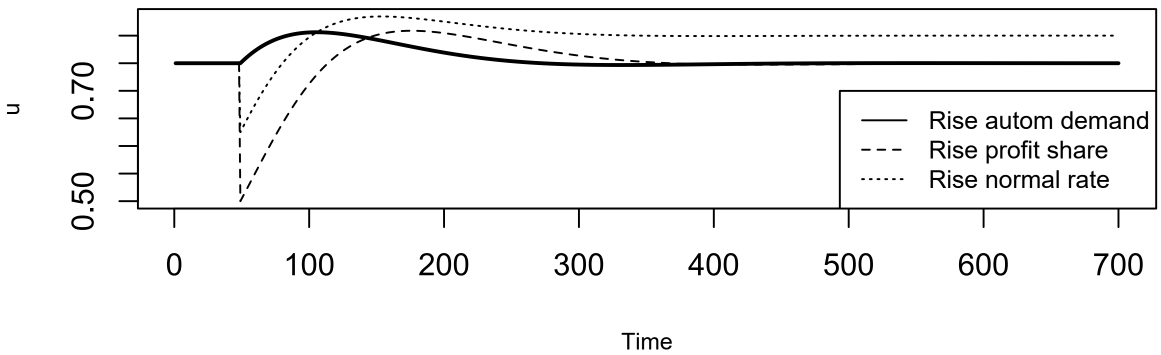


Figure 2: Rate of growth

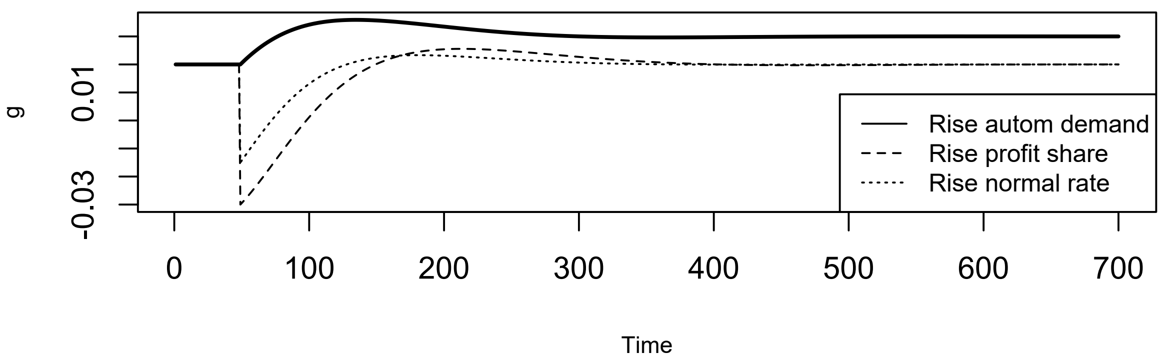
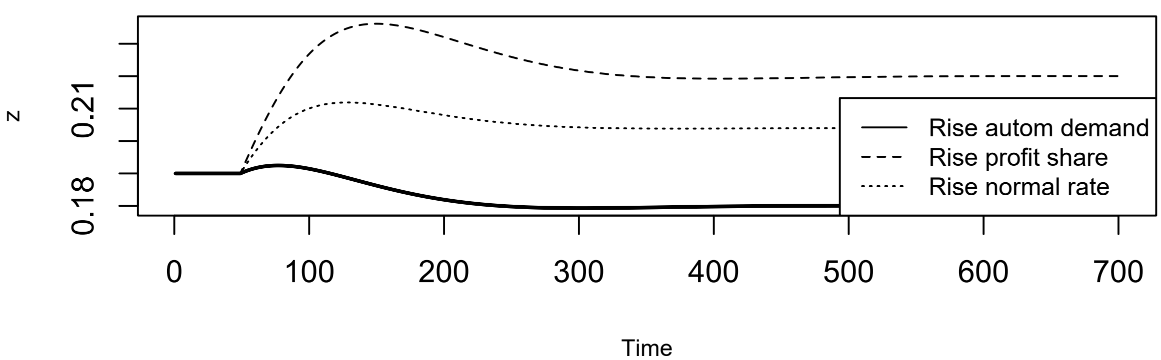


Figure 3: Rate of autonomous demand

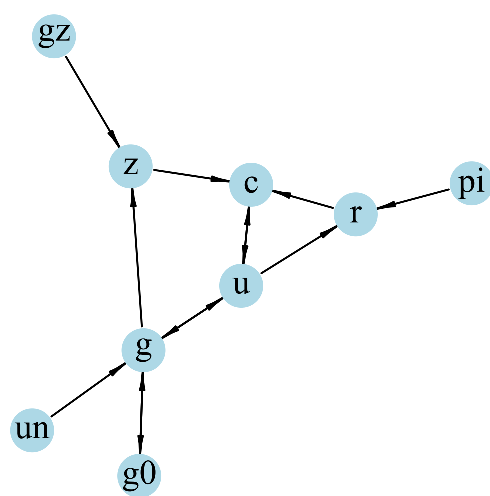


Finally, a rise in the normal rate (dotted line) initially has contractionary effects on utilisation and growth but eventually raises utilisation to a permanently higher level. The growth rate returns to its previous value.

4 Directed graph

Another perspective on the model's properties is provided by its directed graph. A directed graph consists of a set of nodes that represent the variables of the model. Nodes are connected by directed edges. An edge directed from a node x_1 to node x_2 indicates a causal impact of x_1 on x_2.

Figure 4: Directed graph of Sraffian supermultiplier model



In Figure 4, it can be seen that the growth rate of autonomous demand (g_z), the profit share (π), and the normal rate of capacity utilisation (u_n) are the key exogenous variable of the model. The profit rate (r), consumption (c), the autonomous demand rate (z), investment (g), the rate of utilisation (u), an the expected growth rate (g_0) form a closed loop (or cycle) within the system. For example, an increase in the growth rate of autonomous demand increases consumption, which raises the rate of capacity utilisation, growth, and the expected growth rate. In a second-round effect, the increase in the growth rate then feeds back negatively into the autonomous demand rate, which leads to a return of the rate of capacity utilisation to its previous value.

Appendix

A Simulation in continuous time

Equations (6)-(7) are differential equations that describe the change of g_t^0 and z_t in continuous time, i.e. in infinitesimally small time steps. To simulate the model, we approximate these differential equations using the Euler forward method.

Let $\frac{dx}{dt} = G(x)$. We can approximate the continuous change in x by:

$$x(t + \Delta t) = x(t) + G(x(t))\Delta t.$$

As $\Delta t \rightarrow \infty$, this discretised equation becomes equivalent to its counterpart in continuous time, i.e. $G(x)$. In the simulation above, we chose $\Delta t = d = 0.1$.

B Analytical solution

To find the short-run equilibrium solutions for u and g, first substitute (1)-(4) into (5) and solve for u:

$$u^* = \frac{g_0 + z - g_1 u_n}{s_r \pi - g_1}. \tag{8}$$

From this, we get:

$$g^* = g_0 + g_1(u^* - u_n). \tag{9}$$

The long-run equilibrium is given by $u^{**} = u_n$, $g^{**} = g_z$, and (from (7)) $z^{**} = u_n s_r \pi - g_z$.

The dynamics are governed by (6)-(7). The Jacobian matrix is:

$$J(g^0, z) = \begin{bmatrix} \frac{\mu g_1}{s_r\pi - g_1} & \frac{\mu g_1}{s_r\pi - g_1} \\ -z(\frac{g_1}{s_r\pi - g_1} + 1) & \frac{-z g_1}{s_r\pi - g_1} \end{bmatrix}. \tag{10}$$

The determinant of the Jacobian matrix evaluated at the long-run equilibrium is:

$$det(J^*) = \frac{(u_n s_r \pi - g_z)\mu g_1}{s_r \pi - g_1} > 0, \tag{11}$$

which is positive provided $s_r\pi - g_1$, i.e. if the Keynesian stability condition holds.

The trace is:

$$tr(J^*) = \frac{g_1(\mu - u_n s_r \pi + g_z)}{s_r \pi - g_1}. \tag{12}$$

Stability requires a negative trace, yielding a second stability condition: $\mu < u_n s_r \pi - g_z$.

C Construction of directed graph

The directed graph can be derived from the model's Jacobian matrix.[5] Let $\mathbf{x}$ be the vector containing the model's variables.[6] The system of equations representing the model can be written as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The Jacobian matrix is then given by $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$.

Next, construct an 'auxiliary' Jacobian matrix $\mathbf{M}$ in which the non-zero elements of the Jacobian are replaced by ones, whereas zero elements remain unchanged, i.e.

$$M_{ij} = \begin{cases} 1 & \text{if } J_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, taking the transpose of the 'auxiliary' Jacobian matrix yields the adjacency matrix ($\mathbf{M}^\mathbf{T} = \mathbf{A}$), which is a binary matrix whose elements (A_{ji}) indicate whether there is a directed edge from a node x_j to node x_i. From the adjacency matrix, the directed graph is constructed.

[5] See Fennell et al. (2015) for a neat exposition.

[6] Exogenous variables that are supposed to appear in the directed graph can readily be added to the Jacobian by an appropriate extension of rows and columns.

References

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