A Ricardian Two-Sector Model

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1 Overview

This model captures some key feature of David Ricardo's theory of growth and distribution as developed in his 1817 book *On the Principles of Political Economy and Taxation*. The model revolves around the determination of real wages, rents, and profits; and how profitability in turn drives capital accumulation.[1] This version is a two-sector extension of the one-sector model discussed here. It assumes an economy with two sectors: an agricultural sector producing 'corn' subject to diminishing marginal returns and a luxury good sector with constant marginal returns. Prices are determined by the quantity of labour required for production. Rent on the land used for agricultural production is a differential surplus landowners gain based on the fertility of their land relative to the marginal plot of land (the plot of land where fertility is lowest and no rent is earned). Real wages are determined by the subsistence level in the long run. Profits in agriculture are a residual and set the economy-wide profit rate. As employment increases and more land is utilised, marginal productivity in agriculture falls and differential rents increase. As a result, profits are driven down to zero and capital accumulation comes to a halt. A 'stationary state' is reached. Landowners are the main beneficiaries of this process. The model is adapted from Pasinetti (1960).

*I'm grateful to Chandni Dwarkasing for helpful comments. All errors are mine.

[1]See Foley (2006, chap.2) for an excellent introduction.

2 The Model

$$N_t = W_t / w_t \tag{1}$$

$$W_t = K_t \tag{2}$$

$$Y_{1t} = A N_{1t}^{a_1}, \qquad a_1 \in (0, 1) \tag{3}$$

$$N_{1t} = N_t - N_{2t} \tag{4}$$

$$MPL_t = \frac{\partial Y_{1t}}{\partial N_{1t}} = a_1 A N_{1t}^{a_1 - 1} \tag{5}$$

$$R_t = Y_{1t} - N_{1t} MPL_t \tag{6}$$

$$P_{1t} = Y_{1t} - R_t - N_{1t} w_t \tag{7}$$

$$p_{1t} = \frac{1}{MPL_t} \tag{8}$$

$$Y_{2t} = \left(\frac{p_{1t}}{p_2} \right) R_t \tag{9}$$

$$N_{2t} = \frac{Y_{2t}}{a_2} \qquad a_2 > 0 \tag{10}$$

$$p_2 = \frac{1}{a_2} \tag{11}$$

$$P_{2t} = Y_{2t} - \left(\frac{p_{1t}}{p_2} \right) N_{2t} w_t \tag{12}$$

$$P_t = p_{1t} Y_{1t} + p_2 Y_{2t} - p_{1t} R_t - p_{1t} W_t \tag{13}$$

$$K_t = K_{t-1} + g \left(\frac{P_{t-1}}{p_{1t-1}} \right) \tag{14}$$

$$w_t = w_{t-1} - b(w_{t-1} - MPL_{t-1}) \tag{15}$$

where N_t, w_t, W_t, K_t, Y_t, MPL_t, R_t, P_t, and p_t are employment, the real wage (in terms of corn), the wage bill or wage fund (in terms of corn), the capital stock (in terms of corn), real output, the marginal product of labour in sector 1, rents (in terms of corn), profits, and prices, respectively. The subscripts 1 and 2 denote the agricultural and the luxury goods sectors, respectively.

Equation (1) says that total employment is determined by the available wage fund, given the cost of labour. By equation (2), the wage fund is defined as the capital stock (reflecting the fact that production only involves labour). Equation (3) is the production function for corn, exhibiting diminishing marginal returns to labour.[2] By equation (4), employment in

[2] Pasinetti (1960) specifies a generic function $f(N_t)$ with $f(0) \geq 0$, $f'(0) > w^*$, and $f''(N_t) < 0$. Equation

agriculture is residually determined after employment in the luxury goods sector has been determined (more on this below). Equation (5) specifies the marginal product of labour in sector 1, which will be important for the determination of real wages below. Equation (6) captures the determination of (differential) rents as a negative function of the marginal product of labour.[3] Thus, the lower the productivity on the marginal land, the higher the rents. By equation (7), profits in agriculture are determined residually. Equation (8) specifies price determination and captures Ricardo's labour theory of value according to which the value of a good (net of rent) is determined by the quantity of labour required to produce it.[4]

Equation (9) specifies that the production of the luxury good is demand determined. Only landlords consume luxuries and they spend all their income (rent) on luxuries.[5] With production in sector 2 demand determined, employment in sector 2 as given by equation (10) must accommodate based on the production function $Y_{2t} = a_2 N_{2t}$. With employment in sector 2 pinned down in this way and total employment given by the wage fund (by equation (1)), employment in sector 1 must be the residual (as specified in equation (4)). From the labour theory of value, $p_2 Y_2 = N_2$ must hold. Together with the production function $Y_{2t} = a_2 N_{2t}$ this yields equation (11) for the price of the luxury good. Note that due to the constant marginal returns in this sector, its price is constant too.[6] By equation (12), profits in the luxuries sector are determined residually (note that no rent is paid by this sector).

Equation (13) specifies total profits (in nominal terms).[7] Capital accumulation in equation (14) is driven by the reinvestment of profits (with g determining the proportion of profits that are reinvested). Finally, equation (15) specifies real wage dynamics: real wages sluggishly adjust to the level given by the marginal product of labour, which may be interpreted as the subsistence level. If the real wage is below the MPL, demand for labour will increase thereby pushing up real wages.[8]

(3) satisfies these conditions.

[3] Equation (6) is based on the definition of total rent as the sum of the net gains of the non-marginal landowners. See Pasinetti (1960, p.83) for a formal derivation. Note that by using (5), equation (6) can also be written as $R_t = Y_{1t}(1 - a_1)$.

[4] To see this, notice that equation (8) can be derived from $p_{t1}Y_{t1} - p_{t1}R_t = N_{t1}$ if combined with equation (6).

[5] Output in equation (9) is expressed in real terms and can be derived from $p_2 Y_{2t} = p_{1t} R_t$.

[6] The luxury good may therefore serve as Ricardo's 'invariable standard of value' in terms of which the value of all commodities could be expressed.

[7] Note that by combining (13) with (9), total profits can also be written as $P_t = p_{1t}(Y_{1t} - W_t)$. In other words, total profits are independent of output in sector 2.

[8] Equation (15) deviates somewhat from Pasinetti (1960). While Pasinetti introduces dynamic adjustment through employment dynamics, $N_t - N_{t-1} = F(w - MPL)$, our specification instead treats the real wage as a state variable. In line with Pasinetti, our specification ensures that the real wage is equal to the MPL in equilibrium. However, it seems that a problem with Pasinetti's specification is that it generates employment

3 Simulation

Table 1 reports the parameterisation and initial values used in the simulation. In line with the Classical tradition, it will be assumed that all profits are reinvested, i.e. $g = 1$. Besides a baseline (labelled as scenario 1), four further scenarios will be considered. Scenarios 2-4 model three different forms of technological change: an increase in the productivity parameter A (scenario 2), an increase in the elasticity a_1 of agricultural output with respect to labour (scenario 3), and an increase in labour productivity a_2 in the luxury good sector (scenario 4). Scenario 5 considers a higher initial endowment with capital (K_0) that can be used to hire workers. All scenarios initialise employment, the real wage, and capital stock below their steady state values.

Table 1: Parameterisation and initial values

Scenario	A	a_1	a_2	g	b	K_0	N_0	w_0
1: baseline	2	0.7	0.5	1	0.1	1	2	0.1
2: productivity boost I (A)	3	0.7	0.5	1	0.1	1	2	0.1
3: productivity boost II (a_1)	2	0.75	0.5	1	0.1	1	2	0.1
4: productivity boost III (a_2)	2	0.7	0.55	1	0.1	1	2	0.1
5: higher endowment	2	0.7	0.5	1	0.1	5	2	0.1

Figures 1-4 illustrate the model's dynamics under the baseline parameterisation. Starting from below-equilibrium levels, the economy grows in terms of output, capital, and employment but then approaches what Ricardo famously called a 'stationary state'. Figure 3 shows that during the adjustment phase, the MPL declines, reflecting diminishing marginal returns in agriculture. This captures the idea that a growing economy will have to utilise less fertile lands. The real wage is driven up until it is equal to the MPL. Figure 4 shows that total profits initially increase but are then squeezed to zero as differential rents increase.

growth only for real wages exceeding the MPL. But for $w > MPL$, profits will be negative and there will be no capital accumulation. Our specification circumvents this problem by allowing real wages to be below the MPL during the adjustment phase.

Figure 1: Employment (N_t) and capital accumulation (K_t), baseline

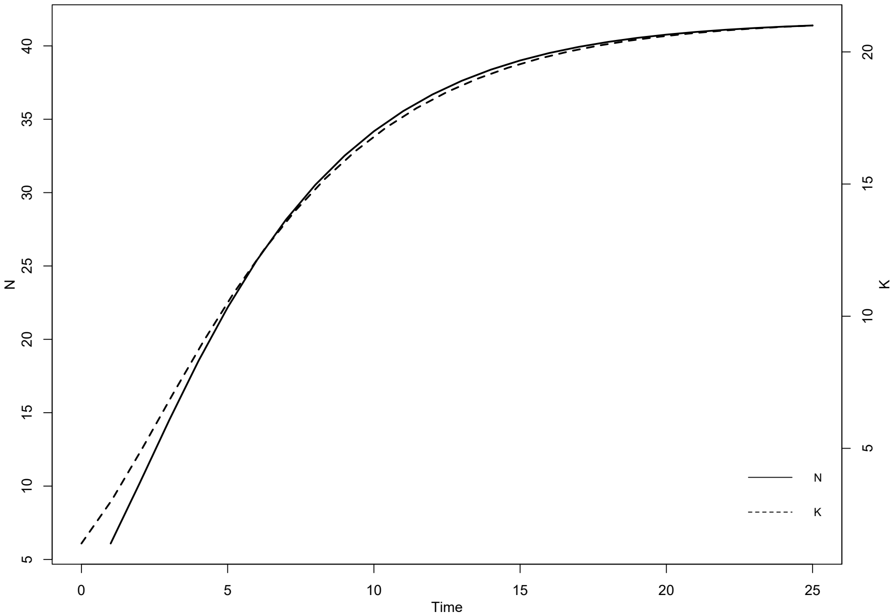


Figure 2: Output in agricultural (Y_{1t}) and luxury goods sector (Y_{2t}), baseline

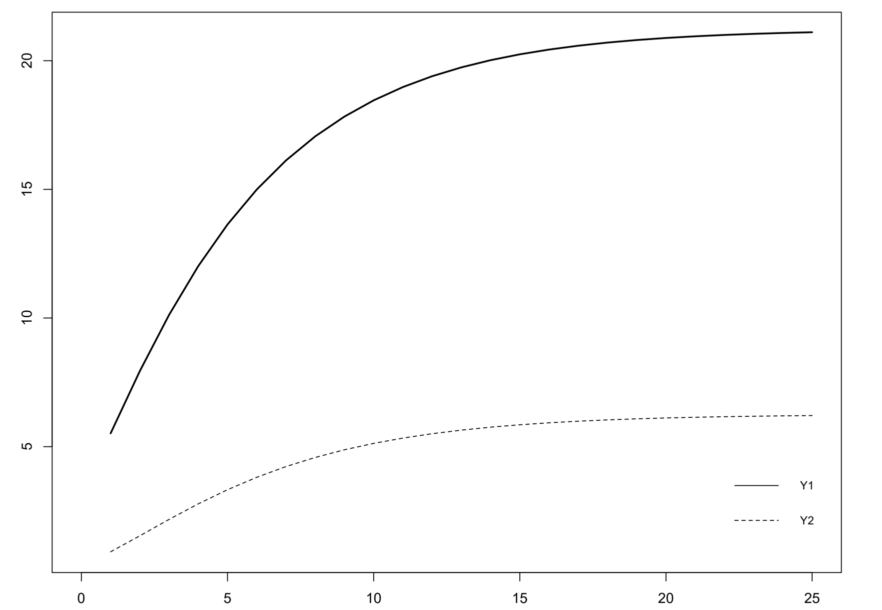


Figure 3: Real wage (w_t) and marginal product of labour (MPL_t), baseline

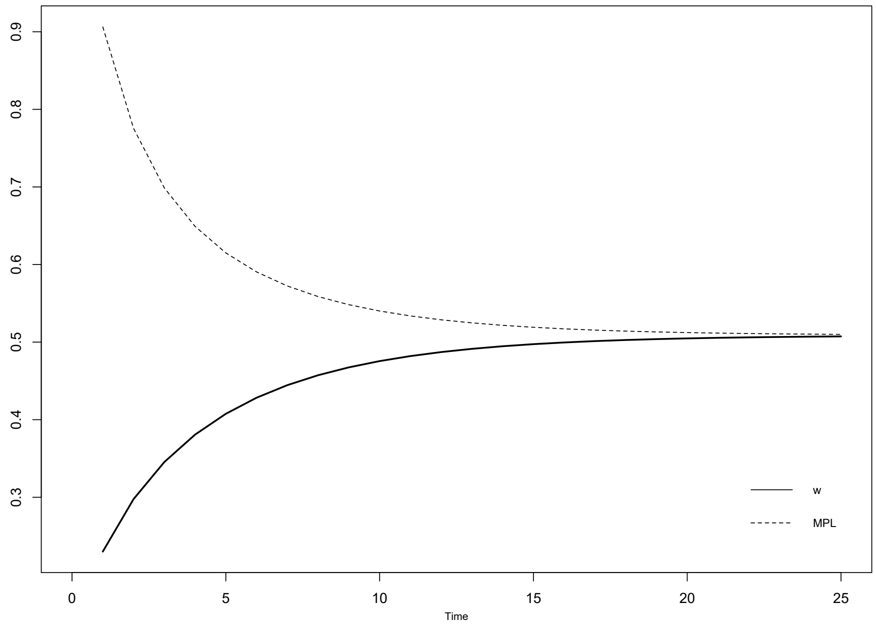
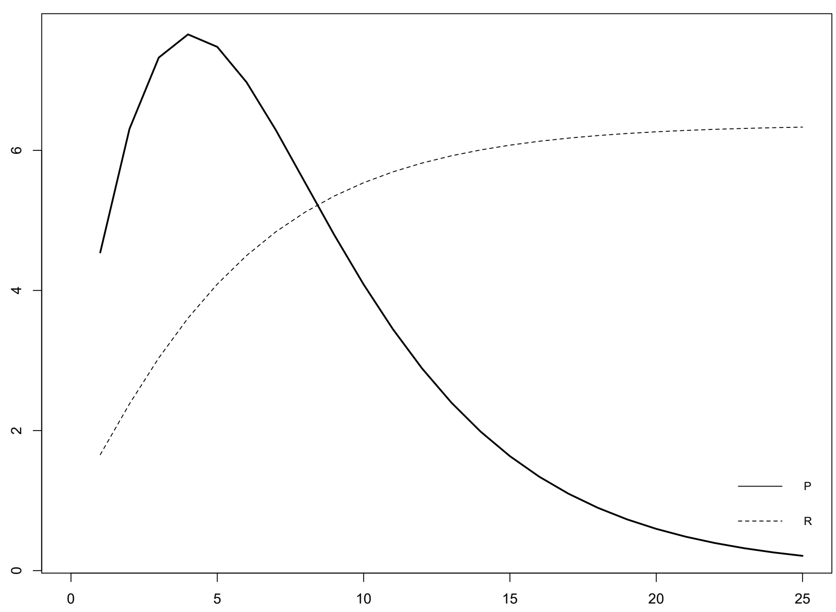
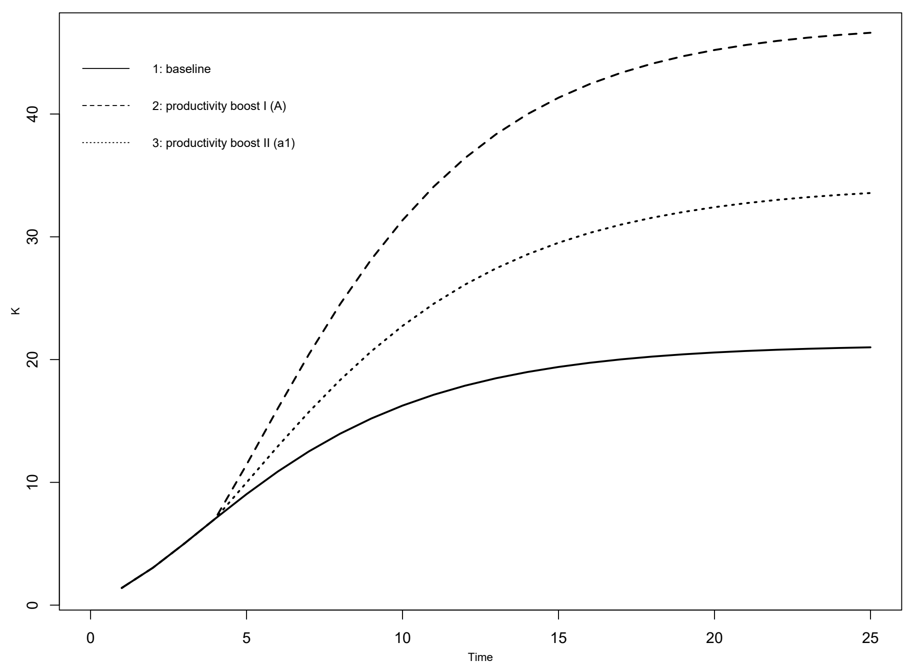


Figure 4: Total profits (P_t) and rents (R_t), baseline



Figures 5-6 display capital accumulation under the five different scenarios described in Table 1. Technical change that increases productivity in agriculture (scenarios 2 and 3) raises the speed of capital accumulation and the equilibrium level of capital. By contrast, an increase in productivity in the luxury good sector (scenario 4) has no effect on capital accumulation. This is because productivity in sector 2 has no effects on functional income distribution.[9] An increase in the initial stock of capital (scenario 5) raises the steady state value. Thus, economies with larger initial endowments will reach a higher level of income in the stationary state.

Figure 5: Capital accumulation (K_t) under baseline and scenarios 2 and 3



[9]The increase in a_2 does raise real output and profits in sector 2 but it leaves total profits unchanged.

Figure 6: Capital accumulation (K_t) under baseline and scenarios 4 and 5

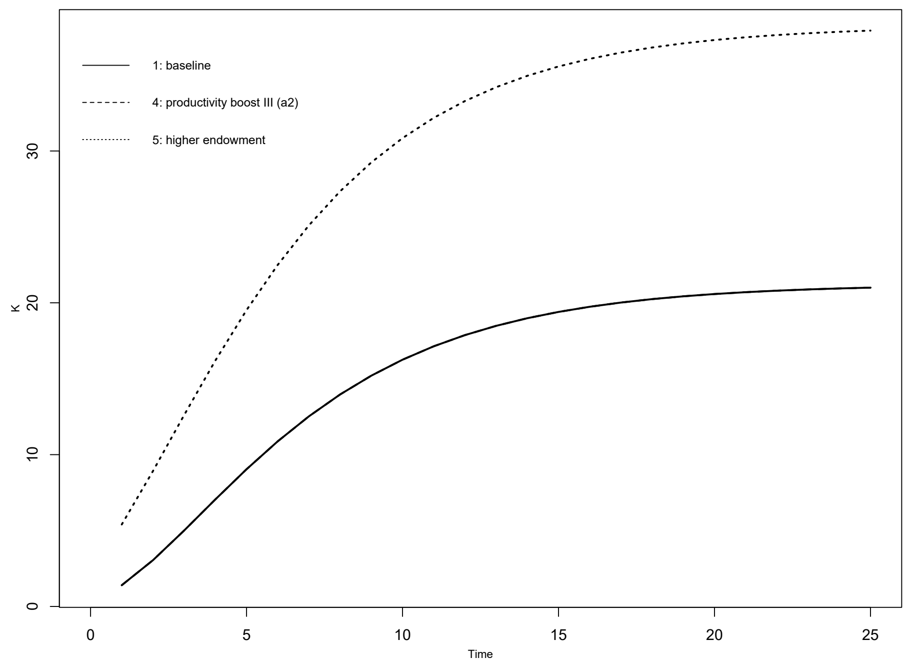


Figure 7 illustrates how changes in productivity affect the composition of output (as measured by the ratio of real output in sector 1 to sector 2). It can be seen that improvements in productivity in one sector raise its output relative to the other sector. Figure 8 shows the dynamics of relative prices (corn price relative to luxury good price). Over time, corn becomes more expensive in relative turns due to diminishing marginal returns. Improvements in labour productivity reduce the relative price of the respective sector.

Figure 7: Output composition ($\frac{Y_{1t}}{Y_{2t}}$) under scenarios 1-4

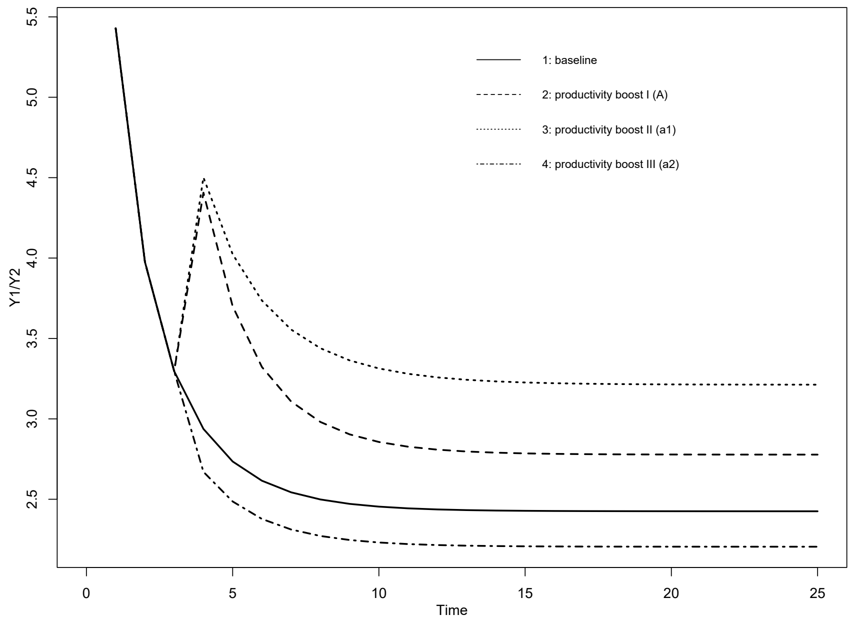
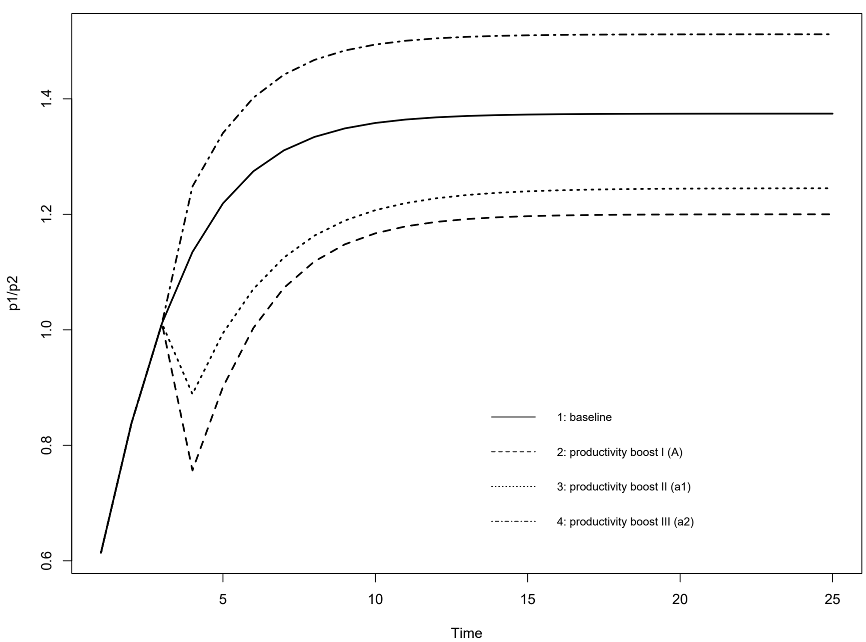


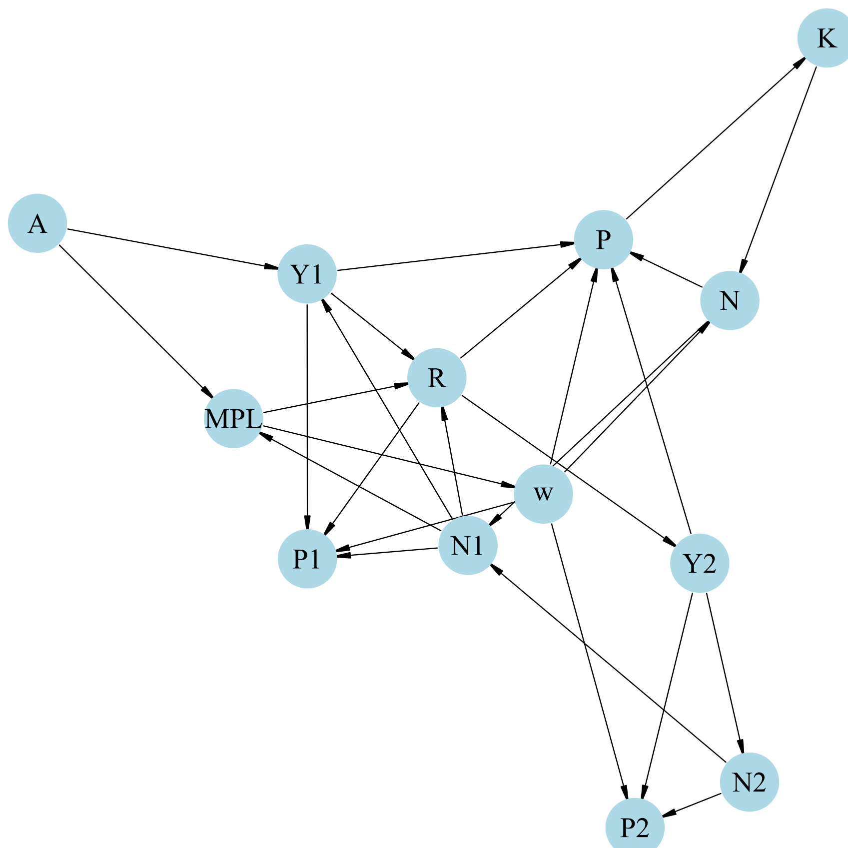
Figure 8: Relative prices ($\frac{p_{1t}}{p_{2t}}$) under scenarios 1-4



4 Directed graph

Another perspective on the model's properties is provided by its directed graph. A directed graph consists of a set of nodes that represent the variables of the model. Nodes are connected by directed edges. An edge directed from a node x_1 to node x_2 indicates a causal impact of x_1 on x_2.

Figure 9: Directed graph of Ricardian Two-Sector Model



Notes: The directed graph depicts the model's steady state. Prices were excluded for clarity.

In Figure 9, it can be seen that productivity (A) is the key exogenous variable that impacts income in sector 1 and the marginal product of labour. All other variables are endogenous and form a closed loop (or cycle) within the system. The directed graph illustrates the supply-driven nature of the agricultural sector, where (marginal) productivity determine employment and distribution. By contrast, the luxury goods sector is demand-determined with employment being the residual. Profits determine capital accumulation, which in turn provides funds that can be used to hire more agricultural workers.

Appendix

A Analytical Discussion

From equations (14) and (15), it can be readily seen that

$$P^* = 0 \tag{16}$$
$$w^* = MPL^*, \tag{17}$$

i.e. profits are zero in equilibrium and the real wage is equal to the marginal product of labour.

Combining equations (1)-(10), substitution into (14) (with $g = 1$) and (15), and some tedious algebra reduces the model to a two-dimensional dynamic system in K_t and w_t:

$$K_t = a_1^{a_1} A \left(\frac{K_{t-1}}{w_{t-1}} \right)^{a_1} \tag{18}$$
$$w_t = w_{t-1}(1-b) + b a_1^{a_1} \left(\frac{K_{t-1}}{w_{t-1}} \right)^{a_1 - 1} \tag{19}$$

The Jacobian matrix is given by:

$$J(K, w) = \begin{bmatrix} a_1^{1+a_1} A K^{a_1-1} w^{-a_1} & -a_1^{1+a_1} A K_1^{a} w^{-1-a_1} \\ bAa_1^{a_1}(a_1-1)K^{a_1-2}w^{1-a_1} & 1-b-b(a_1-1)a_1^{a_1} A K^{a_1-1} w^{-a_1} \end{bmatrix}. \tag{20}$$

From (18), an equation can be derived that characterises the equilibrium relationship between K_t and w_t:

$$K^* = (w^*)^{\frac{-a_1}{1-a_1}} \left(a_1^{a_1} A\right)^{\frac{1}{1-a_1}}. \tag{21}$$

This shows that the steady state capital stock and real wage are inversely related.

We can use this steady-state relationship to derive $a_1^{a_1} A K^{*a_1-1} w^{*-a_1} = 1$. Using this in the Jacobian matrix yields:

$$J^* = \begin{bmatrix} a_1 & -a_1 K w^{-1} \\ b(a_1 - 1)K^{-1} w & 1 - b a_1 \end{bmatrix}. \tag{22}$$

The trace and determinant of the Jacobian matrix at the equilibrium are then given by:

$$Tr(J^*) = a_1 + 1 - ba_1 = a_1(1-b) + 1 \tag{23}$$

$$Det(J^*) = a_1(1 - ba_1) + a_1 b(a_1 - 1) = a_1(1-b). \tag{24}$$

It can be seen that $Det(J^*) = Tr(J^*) - 1$. A well-known property of a matrix's eigenvalues λ_i is $Tr = \lambda_1 + \lambda_2$ and $Det = \lambda_1 \lambda_2$. Thus, if either of the two eigenvalues is unity, $Det = Tr - 1$. From this, we can conclude that one of the eigenvalues of J^* is unity and the other is $a_1(1-b)$. Imposing $a_1(1-b) < 1$, we can derive the stability condition:

$$b > \frac{a_1 - 1}{a_1}. \tag{25}$$

If this condition is satisfied, the system is semi-stable: it converges to the steady state but the latter depends on the initial conditions. Higher endowments (i.e. a higher initial capital stock), will allow for higher equilibrium values of capital and income.[10]

B Construction of directed graph

The directed graph can be derived from the model's Jacobian matrix.[11] Let $\mathbf{x}$ be the vector containing the model's variables. The system of equations representing the model can be written as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The Jacobian matrix is then given by $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$.

Next, construct an 'auxiliary' Jacobian matrix $\mathbf{M}$ in which the non-zero elements of the Jacobian are replaced by ones, whereas zero elements remain unchanged, i.e.

$$M_{ij} = \begin{cases} 1 & \text{if } J_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, taking the transpose of the 'auxiliary' Jacobian matrix yields the adjacency matrix ($\mathbf{M^T} = \mathbf{A}$), which is a binary matrix whose elements (A_{ji}) indicate whether there is a directed edge from a node x_j to node x_i. From the adjacency matrix, the directed graph is constructed.

[10] It can be shown that the unit-root property of the model does not depend on the assumption that $g = 1$ nor on the assumption of sluggish real wage adjustment postulated in equation (15).

[11] See Fennell et al. (2015) for a neat exposition.

References

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