A Ricardian One-Sector Model

*Karsten Kohler**

Leeds University Business School, k.kohler@leeds.ac.uk.

https://karstenkohler.com

April 2022

Version 2.0

1 Overview

This model captures some key feature of David Ricardo's theory of growth and distribution as developed in his 1817 book *On the Principles of Political Economy and Taxation*. The theory revolves around the determination of real wages, rents, and profits, and how profitability in turn drives capital accumulation.[1] It assumes a corn economy with a single good ('corn') that serves both as an investment and consumption good.[2] Corn production is subject to diminishing marginal returns. Real wages are driven down to a subsistence level and rent is a differential surplus landowners gain based on the fertility of their land relative to the marginal plot of land (the plot of land where fertility is lowest and no rent is earned). Profits are a residual. As employment increases and more land is utilised, marginal productivity falls and differential rents increase. As a result, profits are driven down and capital accumulation comes to a halt. A 'stationary state' is reached. Landowners are the main beneficiaries of this process. The model is adapted from Pasinetti (1960).

*I'm grateful to Chandni Dwarkasing for helpful comments. All errors are mine.

[1]See Foley (2006, chap.2) for an excellent introduction.

[2]A two-sector extension of the model can be found here.

2 The Model

$$
\begin{aligned}
&N_t = W_t/w_t && && (1)\\
&W_t = K_t && && (2)\\
&Y_t = AN_t^a, && a \in (0,1) && (3)\\
&MPL_t = \frac{\partial Y_t}{\partial N_t} = aAN_t^{a-1} && && (4)\\
&R_t = Y_t - N_t MPL_t && && (5)\\
&P_t = Y_t - R_t - N_t w_t && && (6)\\
&K_t = K_{t-1} + gP_{t-1} && && (7)\\
&w_t = w_{t-1} - b(w_{t-1} - MPL_{t-1}) && && (8)
\end{aligned}
$$

where N_t, w_t, W_t, K_t, Y_t, MPL_t, R_t, and P_t are employment, the real wage, the wage bill (or wage fund), the capital stock, output (measured in units of corn), the marginal product of labour, rents, and profits, respectively.

Equation (1) says that total employment is determined by the available wage fund, given the cost of labour. By equation (2), the wage fund is defined as the capital stock of this model (reflecting the fact that the production of corn only involves labour). Equation (3) is the production function exhibiting diminishing marginal returns to labour.[3] Equation (4) specifies the marginal product of labour, which will be important for the determination of real wages below. Equation (5) captures the determination of (differential) rents as a negative function of the marginal product of labour.[4] Thus, the lower the productivity on the marginal land, the higher the rents. In equation (6), profits are determined residually. Capital accumulation in equation (7) is driven by the reinvestment of profits (with g determining the proportion of profits that are reinvested). Finally, equation (8) specifies real wage dynamics: real wages sluggishly adjust to the level given by the marginal product of labour, which may be interpreted as the subsistence level. If the real wage is below the MPL, demand for labour will increase thereby pushing up real wages.[5]

[3] Pasinetti (1960) specifies a generic function $f(N_t)$ with $f(0) \geq 0$, $f'(0) > w^*$, and $f''(N_t) < 0$. Equation (3) satisfies these conditions.

[4] Equation (5) is based on the definition of total rent as the sum of the net gains of the non-marginal landowners. See Pasinetti (1960, p.83) for a formal derivation.

[5] Equation (8) deviates somewhat from Pasinetti (1960). While Pasinetti introduces dynamic adjustment through employment dynamics, $N_t - N_{t-1} = F(w - MPL)$, our specification instead treats the real wage as a state variable. In line with Pasinetti, our specification ensures that the real wage is equal to the MPL in equilibrium. However, it seems that a problem with Pasinetti's specification is that it generates employment

3 Simulation

Table 1 reports the parameterisation and initial values used in the simulation. In line with the Classical tradition, it will be assumed that all profits are reinvested, i.e. $g = 1$. Besides a baseline (labelled as scenario 1), three further scenarios will be considered. Scenarios 2 and 3 model two different forms of technological change: an increase in the productivity parameter A and an increase in the elasticity of output with respect to labour (a). Scenario 4 considers a higher initial endowment with capital (K_0) that can be used to hire workers. All scenarios initialise employment, real wage, and capital stock below their steady state values.

Table 1: Parameterisation and initial values

Scenario	A	a	g	b	N_0	w_0	K_0
1: baseline	2	0.7	1	0.1	2	0.1	1
2: productivity boost I (A)	3	0.7	1	0.1	2	0.1	1
3: productivity boost II (a)	2	0.75	1	0.1	2	0.1	1
4: higher endowment (K_0)	2	0.7	1	0.1	2	0.1	5

Figure 1 displays employment, capital accumulation, and income for the baseline scenario. Starting from a below-equilibrium level of population, real wages, and capital, the economy grows in terms of output, capital, and employment but then approaches what Ricardo famously called a 'stationary state'. Figure 2 shows that during the adjustment phase, the MPL declines reflecting diminishing marginal returns. This captures the idea that a growing economy will have to utilise less fertile lands. The real wage is driven up until it is equal to the MPL. Figure 3 shows that profits initially increase but are then squeezed to zero as differential rents increase.

growth only for real wages exceeding the MPL. But for $w > MPL$, profits will be negative and there will be no capital accumulation. Our specification circumvents this problem by allowing real wages to be below the MPL during the adjustment phase.

Figure 1: Employment (N_t), capital accumulation (K_t), and income (Y_t), baseline

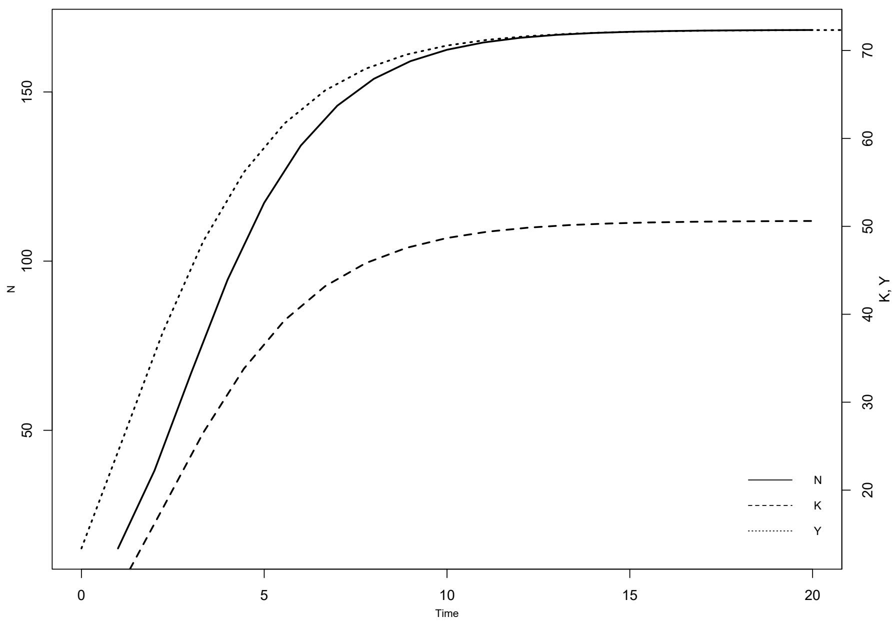


Figure 2: Real wage (w_t) and marginal product of labour (MPL_t), baseline

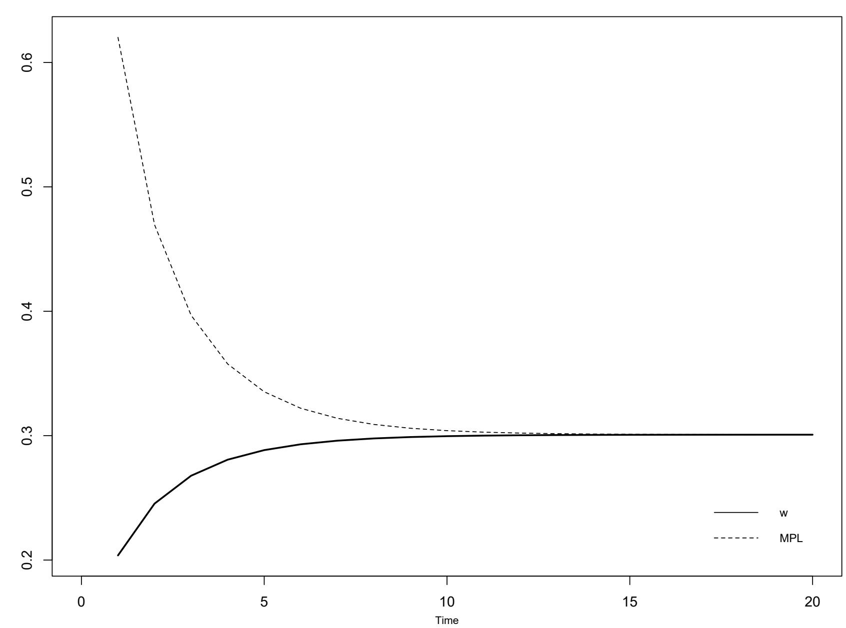


Figure 3: Profits (P_t) and rents (R_t), baseline

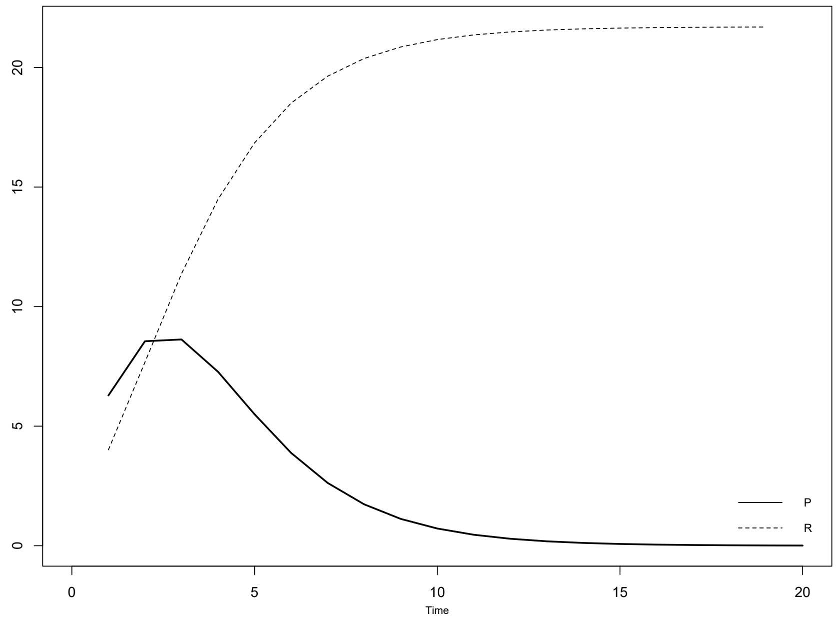
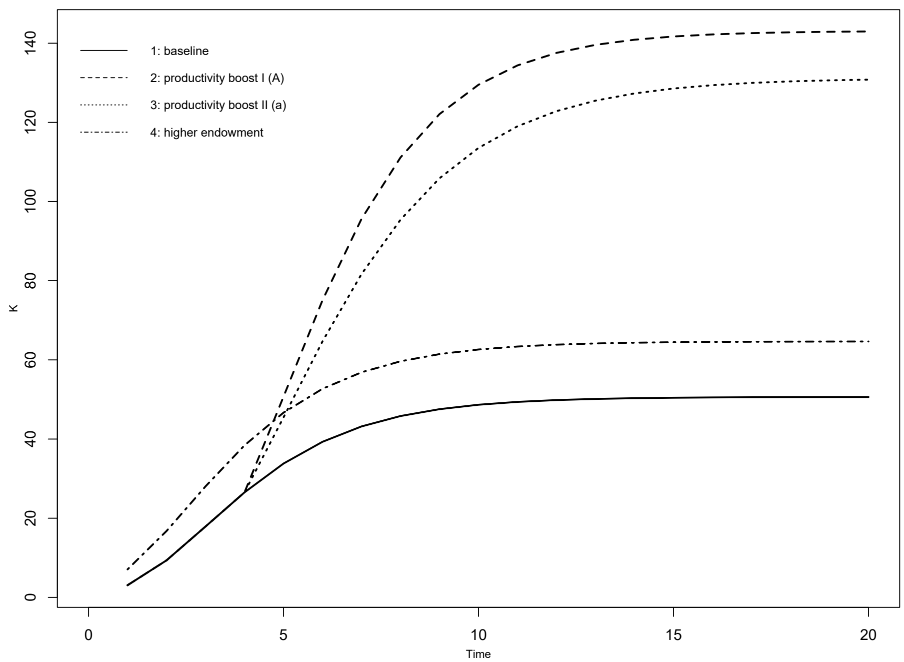


Figure 4 displays capital accumulation under the different scenarios described in Table 1. As expected, both forms of technical change boost both the speed of capital accumulation and the equilibrium level of capital. An increase in the initial stock of capital does not change the pace of capital accumulation but raises the steady state value. Thus, economies with larger initial endowments will reach a higher level of income in the stationary state.

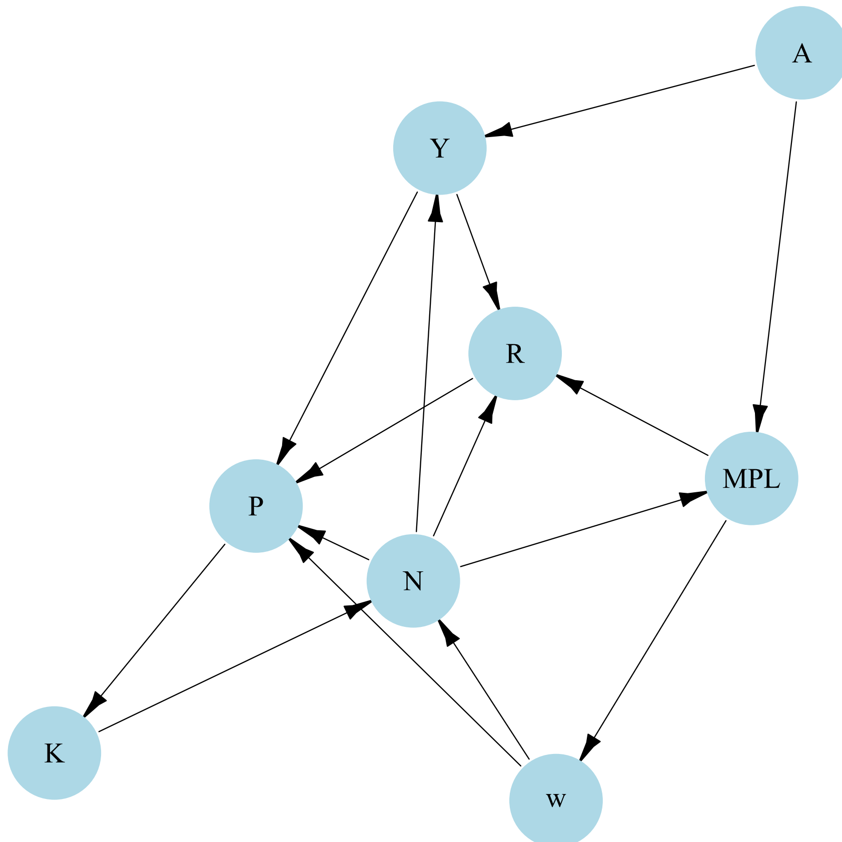
Figure 4: Capital accumulation (K_t) under different scenarios



4 Directed graph

Another perspective on the model's properties is provided by its directed graph. A directed graph consists of a set of nodes that represent the variables of the model. Nodes are connected by directed edges. An edge directed from a node x_1 to node x_2 indicates a causal impact of x_1 on x_2.

Figure 5: Directed graph of Ricardian One-Sector Model



Notes: The directed graph depicts the model's steady state.

In Figure 5, it can be seen that productivity (A) is the key exogenous variable that impacts income and the marginal product of labour. All other variables are endogenous and form a closed loop (or cycle) within the system. The directed graph illustrates the supply-driven nature of the model, where (marginal) productivity determine employment and distribution, which in turn feed back into income and capital accumulation.

Appendix

A Analytical Discussion

From equations (7) and (8), it can readily be seen that

$$P^* = 0 \tag{9}$$
$$w^* = MPL^*, \tag{10}$$

i.e. profits are zero in equilibrium and the real wage is equal to the marginal product of labour. Combining equations (1)-(6) and substitution into (7) (with $g = 1$) as well as using equations (4) and (1) in equation (8) reduces the model to a two-dimensional dynamic system in K_t and w_t:

$$K_t = aA \left(\frac{K_{t-1}}{w_{t-1}} \right)^a \tag{11}$$
$$w_t = w_{t-1}(1-b) + baA \left(\frac{K_{t-1}}{w_{t-1}} \right)^{a-1} \tag{12}$$

The Jacobian matrix is given by:

$$J(K, w) = \begin{bmatrix} a^2 A K^{a-1} w^{-a} & -a^2 A K^a w^{-1-a} \\ bAa(a-1)K^{a-2}w^{1-a} & 1 - b - b(a-1)aAK^{a-1}w^{-a} \end{bmatrix}. \tag{13}$$

From (11), an equation can be derived that characterises the equilibrium relationship between K_t and w_t:

$$K^* = (w^*)^{\frac{-a}{1-a}} (aA)^{\frac{1}{1-a}}. \tag{14}$$

This shows that the steady state capital stock and real wage are inversely related.

We can use this steady-state relationship to derive $aAK^{*a-1}w^{*-a} = 1$. Using this in the Jacobian matrix yields:

$$J^* = \begin{bmatrix} a & -aKw^{-1} \\ b(a-1)K^{-1}w & 1 - ba \end{bmatrix}. \tag{15}$$

The trace and determinant of the Jacobian matrix at the equilibrium are then given by:

$$Tr(J^*) = a + 1 - ba = a(1-b) + 1 \tag{16}$$

$$Det(J^*) = a(1-ba) + ab(a-1) = a(1-b). \tag{17}$$

It can be seen that $Det(J^*) = Tr(J^*) - 1$. A well-known property of a matrix's eigenvalues λ_i is $Tr = \lambda_1 + \lambda_2$ and $Det = \lambda_1 \lambda_2$. Thus, if either of the two eigenvalues is unity, $Det = Tr - 1$. From this, we can conclude that one of the eigenvalues of J^* is unity and the other is $a(1-b)$. Imposing $a(1-b) < 1$, we can derive the stability condition:

$$b > \frac{a-1}{a}. \tag{18}$$

If this condition is satisfied, the system is semi-stable: it converges to the steady state but the latter depends on the initial conditions. Higher endowments (i.e. a higher initial capital stock), will allow for higher equilibrium values of capital and income.[6]

B Construction of directed graph

The directed graph can be derived from the model's Jacobian matrix.[7] Let $\mathbf{x}$ be the vector containing the model's variables. The system of equations representing the model can be written as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The Jacobian matrix is then given by $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$.

Next, construct an 'auxiliary' Jacobian matrix $\mathbf{M}$ in which the non-zero elements of the Jacobian are replaced by ones, whereas zero elements remain unchanged, i.e.

$$M_{ij} = \begin{cases} 1 & \text{if } J_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, taking the transpose of the 'auxiliary' Jacobian matrix yields the adjacency matrix ($\mathbf{M^T} = \mathbf{A}$), which is a binary matrix whose elements (A_{ji}) indicate whether there is a directed edge from a node x_j to node x_i. From the adjacency matrix, the directed graph is constructed.

[6]It can be shown that the unit-root property of the model does not depend on the assumption that $g = 1$ nor on the assumption of sluggish real wage adjustment postulated in equation (8).

[7]See Fennell et al. (2015) for a neat exposition.

References

Fennell, P. G., O'Sullivan, D. J. P., Godin, A. & Kinsella, S. (2015), 'Is it possible to visualise any stock flow consistent model as a directed acyclic graph?', *Computational Economics* **48**(2), 307–316.

Foley, D. K. (2006), *Adam's Fallacy. A Guide to Economic Theology*, Harvard University Press, Cambridge, MA / London.

Pasinetti, L. L. (1960), 'A Mathematical Formulation of the Ricardian System', *The Review of Economic Studies* **27**(2), 78–98.