A Malthusian Model

Karsten Kohler

Leeds University Business School, k.kohler@leeds.ac.uk.

https://karstenkohler.com

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1 Overview

This model captures some key feature of Thomas Malthus' theory of population dynamics as developed in his 1798 book *An Essay on the Principle of Population*. The theory revolves around the interaction between living standards and population growth.[1] It assumes that birth rates increase with rising living standards, while death rates decline. Economic growth thus spurs population growth. However, due to supply constraints in agricultural production, population growth drives up food prices and thereby undermines real income, bringing population growth to a halt. The model is adapted from Karl Whelan's lecture notes.

2 The Model

$$N_t = N_{t-1} + B_{t-1} - D_{t-1} \tag{1}$$

$$\frac{B_t}{N_t} = b_0 + b_1 Y_t \tag{2}$$

$$\frac{D_t}{N_t} = d_0 - d_1 Y_t \tag{3}$$

$$Y_t = a_0 - a_1 N_t, \tag{4}$$

where N_t, B_t, D_t, and Y_t are population, number of births, number of deaths, and real income, respectively.

Equation (1) describes population dynamics as driven by births and deaths. Equations (2)-(3) capture the Malthusian hypothesis that birth rates are positively and death rates

[1]See Foley (2006, chap.2) for an excellent introduction.

negatively related to income. Equation (4) makes real income a negative function of the population, which captures the idea of supply constraints in agriculture.

3 Simulation

Table 1 reports the parameterisation and initial values used in the simulation. Besides a baseline (labelled as scenario 1), three further scenarios will be considered. Scenario 2 models what Malthus called 'preventative checks': a fall in the exogenous component of the birth rate (b_0) due to an increased use of contraception, changes in marriage norms etc. Scenario 3 models 'positive checks': a rise in the sensitivity of real income with respect to the population (a_1), capturing factors such as increased food scarcity. Scenario 4 considers a rise in the exogenous component of real income (a_0), which could be interpreted as a productivity boost due to the invention of better fertilisers. All scenarios initialise the population below its steady state value at $N_0 = 1$ and the other variables at their steady state values.

Table 1: Parameterisation, initial values, and steady state values

Scenario	b_0	b_1	d_0	d_1	a_0	a_1	N_0	N^*	Y_0	Y^*	B_0	B^*	D_0	D^*
1: baseline	0.5	0.5	2.5	0.5	2.5	0.05	1	10	Y^*	2	B^*	15	D^*	15
2: fall in exog birth rate (b_0)	0.4	0.5	2.5	0.5	2.5	0.05	1	8	Y^*	2.1	B^*	11.6	D^*	11.6
3: rise in sensitivity of income (a_1)	0.5	0.5	2.5	0.5	2.5	0.07	1	7.1	Y^*	2	B^*	10.7	D^*	10.7
4: productivity boost (a_0)	0.5	0.5	2.5	0.5	2.6	0.05	1	12	Y^*	2	B^*	18	D^*	18

Figure 1 displays population and real income dynamics for the baseline scenario. Starting from a below-equilibrium level of population, the population initially grows rapidly (seemingly exponentially) but then approaches a steady state. During the adjustment phase, real income is driven down to its steady state level (which can be interpreted as the subsistence level). Figure 2 displays the corresponding dynamics of births and deaths.

Figure 1: Population (N_t) and real income (Y_t), baseline

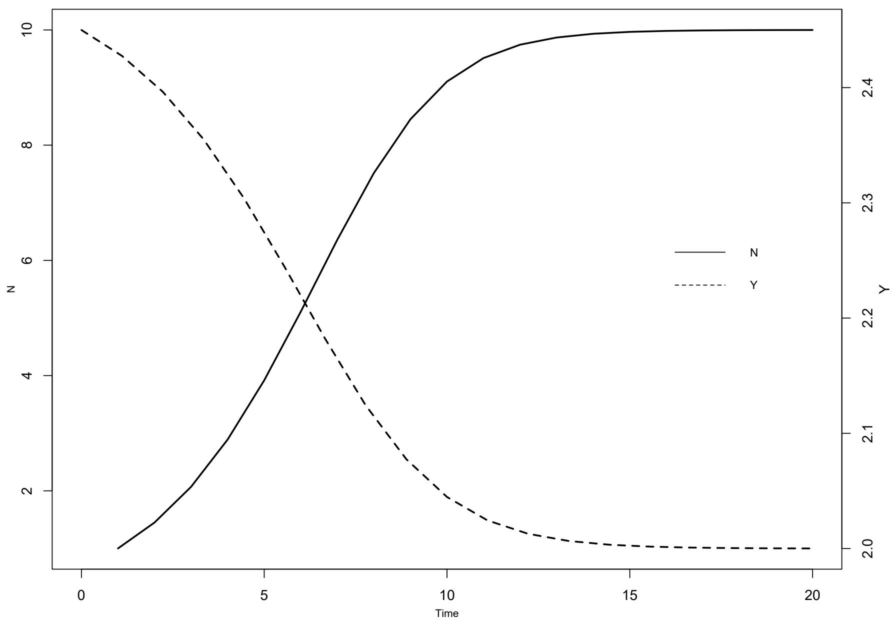


Figure 2: Births (B_t) and deaths (D_t), baseline

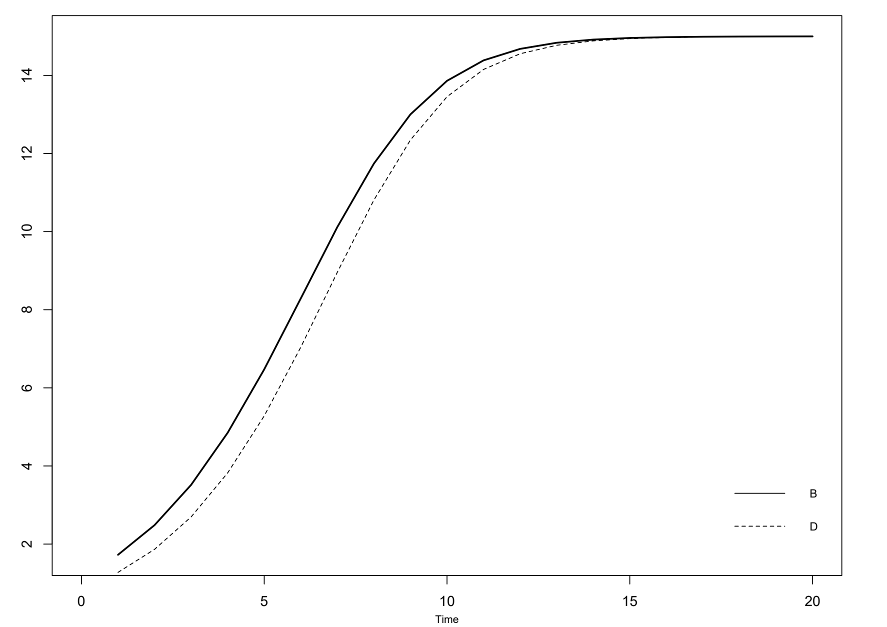
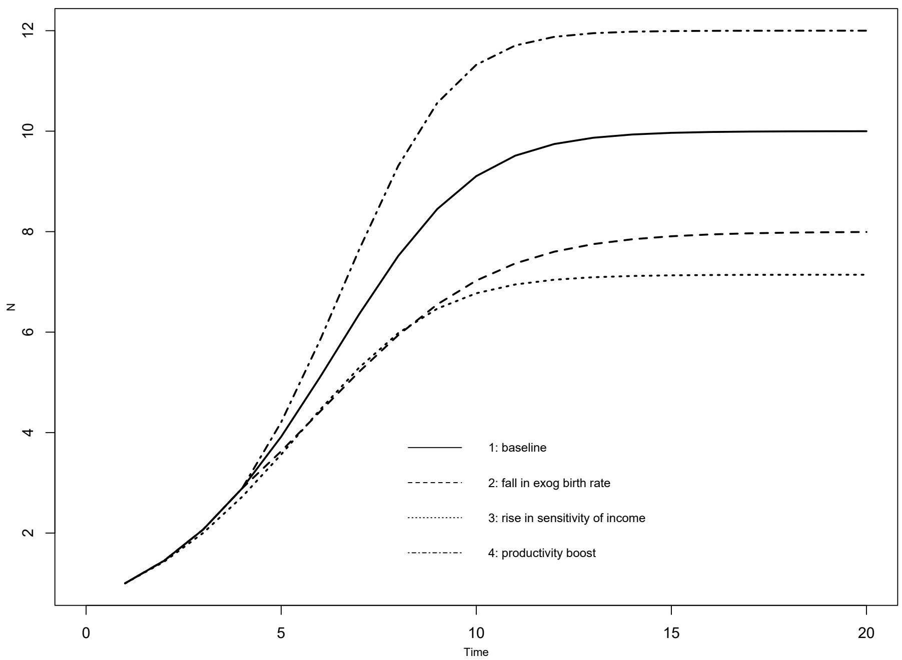


Figure 3 displays population dynamics under the different scenarios described in Table 1.

As expected, both preventative and positive checks are effective: a fall in the exogenous component of the birth rate and an increase in the sensitivity of real income slow down population dynamics and lower its steady state value. By contrast, a productivity boost allows for a higher equilibrium level of population.

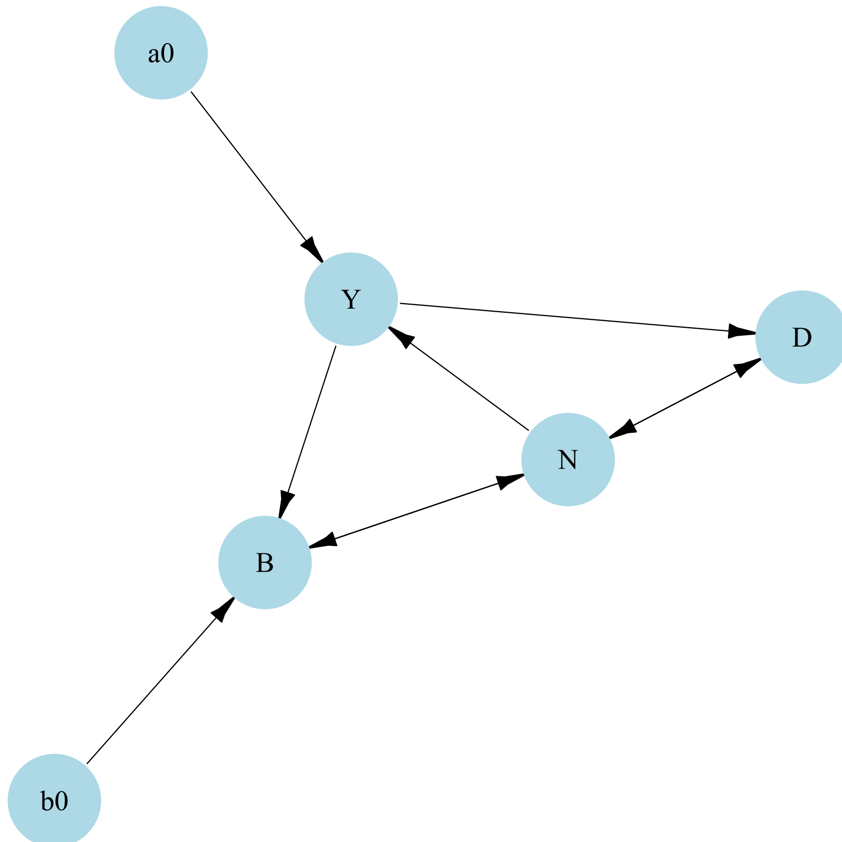
Figure 3: Population dynamics (N_t) under different scenarios



4 Directed graph

Another perspective on the model's properties is provided by its directed graph. A directed graph consists of a set of nodes that represent the variables of the model. Nodes are connected by directed edges. An edge directed from a node x_1 to node x_2 indicates a causal impact of x_1 on x_2.

Figure 4: Directed graph of Malthusian Model



Notes: The directed graph depicts the model's steady state.

In Figure 4, it can be seen that the exogenous birth rate ($b0$) and productivity ($a0$) are exogenous variables that impact births and income, respectively. Births, deaths, employment and income are endogenous and form a closed loop (or cycle) within the system. Births and deaths affect the population size (with simultaneous feedback from population to births and deaths), and the population affects income. Income, in turn, feeds back into population size.

Appendix

A Analytical Solution

To find the steady state solution for N, substitute (2)-(4) into (1) and collect terms:

$$N_t = N_{t-1}[1 + b_0 - d_0 + a_0(b_1 + d_1)] - N_{t-1}^2[a_1(b_1 + d_1)]. \tag{5}$$

Subtract N_{t-1} and divide through by N_{t-1}:

$$\frac{N_t - N_{t-1}}{N_{t-1}} = [b_0 - d_0 + a_0(b_1 + d_1)] - N_{t-1}[a_1(b_1 + d_1)]. \tag{6}$$

Set $\frac{N_t - N_{t-1}}{N_t t-1} = 0$ and solve for N_t to find the non-trivial steady state:[2]

$$N^* = \frac{b_0 - d_0 + a_0(b_1 + d_1)}{a_1(b_1 + d_1)}. \tag{7}$$

Substitution of N^* into (4) and simplifying yields:

$$Y^* = \frac{d_0 - b_0}{b_1 + d_1}. \tag{8}$$

Finally, to assess the dynamic stability of the model, differentiate (5) with respect to N_{t-1}:

$$\frac{\partial N_t}{\partial N_{t-1}} = 1 + b_0 - d_0 + a_0(b_1 + d_1) - 2N_{t-1}[a_1(b_1 + d_1)]. \tag{9}$$

Due to then non-linearity of the model, stability can only be assessed locally around the steady state. To do this, substitute the steady state solution and simplify:

$$\frac{\partial N_t}{\partial N_{t-1}} = 1 - b_0 + d_0 - a_0(b_1 + d_1). \tag{10}$$

[2]A trivial steady state is at $N^* = 0$.

From this, we can conclude that the steady state is stable iff:

$$|1 - b_0 + d_0 - a_0(b_1 + d_1)| < 1. \tag{11}$$

B Construction of directed graph

The directed graph can be derived from the model's Jacobian matrix.[3] Let $\mathbf{x}$ be the vector containing the model's variables. The system of equations representing the model can be written as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The Jacobian matrix is then given by $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$.

Next, construct an 'auxiliary' Jacobian matrix $\mathbf{M}$ in which the non-zero elements of the Jacobian are replaced by ones, whereas zero elements remain unchanged, i.e.

$$M_{ij} = \begin{cases} 1 & \text{if } J_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, taking the transpose of the 'auxiliary' Jacobian matrix yields the adjacency matrix ($\mathbf{M}^{\mathbf{T}} = \mathbf{A}$), which is a binary matrix whose elements (A_{ji}) indicate whether there is a directed edge from a node x_j to node x_i. From the adjacency matrix, the directed graph is constructed.

[3]See Fennell et al. (2015) for a neat exposition.

References

Fennell, P. G., O'Sullivan, D. J. P., Godin, A. & Kinsella, S. (2015), 'Is it possible to visualise any stock flow consistent model as a directed acyclic graph?', *Computational Economics* **48**(2), 307–316.

Foley, D. K. (2006), *Adam's Fallacy. A Guide to Economic Theology*, Harvard University Press, Cambridge, MA / London.