A New Keynesian 3-Equation Model

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1 Overview

New Keynesian dynamic general equilibrium models were developed in the 1990s and 2000s to guide monetary policy.[1] They build on real business cycle models with rational expectations but introduce Keynesian frictions such as imperfect competition and nominal rigidities. While the structural forms of these models are typically complex as behavioural functions are derived from the intertemporal optimisation, the reduced-form of the benchmark models can be represented by three main equations: (i) an IS curve, (ii) a Phillips curve, (iii) and an interest rate rule.

The IS curve establishes a negative relationship between real income and the real interest rate. For a higher real interest rate, households will save more and thus consume less. The Phillips curve models inflation as a function of the output gap. A positive output gap (an economic expansion) leads to higher inflation. The monetary policy rule specifies how the central bank reacts to deviations of actual inflation from a politically determined inflation target.

The simplified version of the 3-equation model we consider here is directly taken from Carlin & Soskice (2014, chap. 3). This is a short-run model in which prices are flexible but the capital stock is fixed. The focus is thus on goods market equilibrium rather than economic growth. In the Carlin-Soskice version, inflation expectations are assumed to be adaptive and the response of aggregate demand to a change in the interest rate is sluggish. This renders the model dynamic.[2]

[1] See Galí 2018 for an overview.

[2] Note that this is quite different from conventional New Keynesian dynamic general equilibrium models in which the dynamic element stems from the rational expectations of agents. Sluggish adjustment in these models is generated by serially correlated shocks.

2 The Model

$$y_t = A - a_1 r_{t-1}, \qquad a_1 > 0 \tag{1}$$

$$\pi_t = \pi_{t-1} + a_2(y_t - y_e), \qquad a_2 > 0 \tag{2}$$

$$r_s = \frac{(A - y_e)}{a_1} \tag{3}$$

$$r_t = r_s + a_3(\pi_t - \pi^T), \qquad a_3 = \frac{1}{a_1(\frac{1}{a_2 b} + a_2)} > 0 \tag{4}$$

where y, A, r, π, y_e, r_s, and π^T are real output, autonomous demand (times the multiplier), the real interest rate, inflation, equilibrium output, the stabilising real interest rate, and the inflation target, respectively.

Equation (1) is the IS curve or goods market equilibrium condition. Aggregate output (y) adjusts to the level of aggregate demand, which is given by autonomous demand (times the multiplier) and a component that is negatively related to the (lagged) real interest rate via households' saving. Equation (2) is the Phillips curve. It is assumed that inflation is driven by adaptive expectations ($E[\pi_{t+1}] = \pi_{t-1}$) as well as the output gap ($y_t - y_e$). By equation (3), the stabilising real interest rate (r_s) is that real interest rate that is consistent with equilibrium output ($y_e = A - a_1 r_s$). Finally, the interest rate rule in (4) specifies the real interest rate the central bank needs to set to minimise its loss function.[3]

3 Simulation

Table 1 reports the parameterisation used in the simulation. For all parameterisations, the system is initialised at the equilibrium $(y^*, \pi^*, r^*) = (y_e, \pi^T, r_s)$. Three scenarios will then be considered. In scenario 1, there is an increase in autonomous aggregate demand (A). In scenario 2, the central bank sets a higher inflation target (π^T). Scenario 3 considers a rise in equilibrium output (y_e).

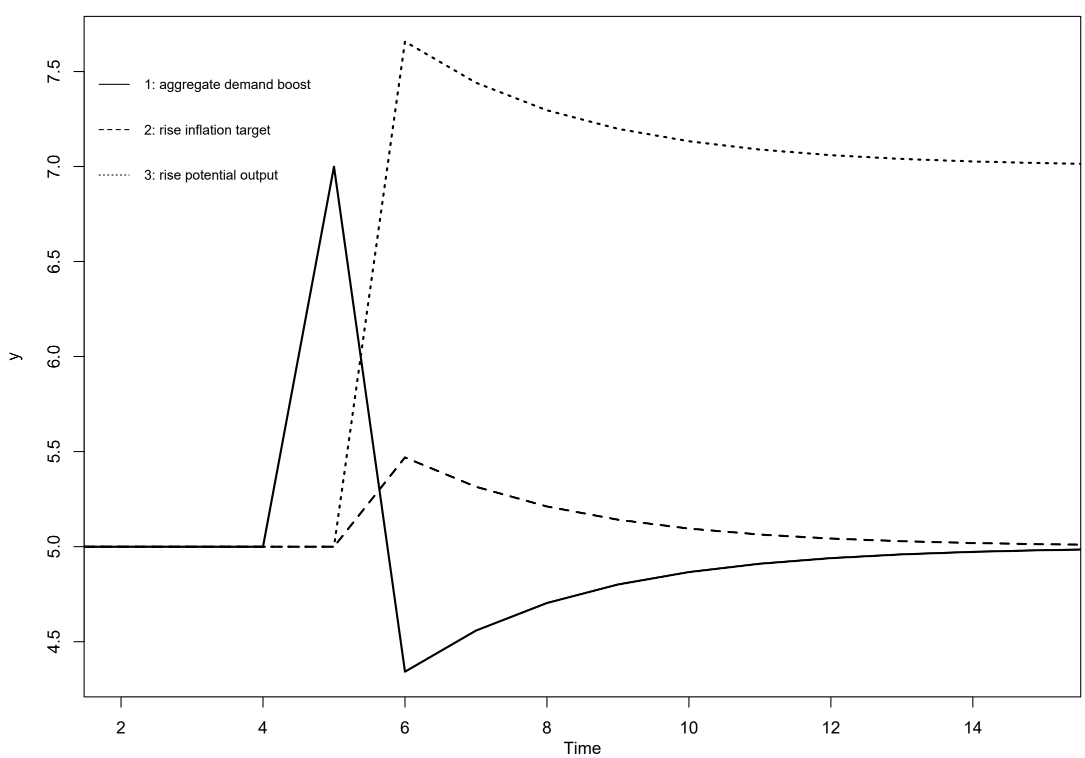
[3]While the central bank only sets the nominal interest rate $i = r + E[\pi_{t+1}]$ directly, the fact that expected inflation is predetermined in every period allows it to indirectly control the real interest rate. See the Appendix for a more detailed derivation of these equations.

Table 1: Parameterisation

Scenario	a_1	a_2	b	A	π^T	y_e
1: rise in aggregate demand (A)	0.3	0.7	1	12	2	5
2: higher inflation target (π^T)	0.3	0.7	1	10	2.5	5
3: rise in equilibrium output (y_e)	0.3	0.7	1	10	2	7

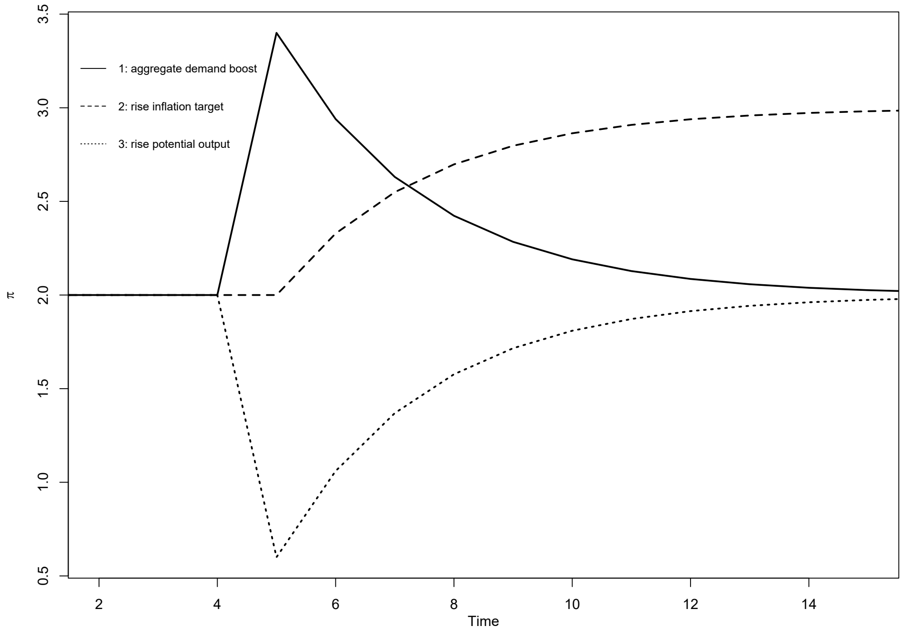
Figures 1-3 depict the response of the model's key endogenous variables to various shifts. A permanent rise in aggregate demand (scenario 1) has an instantaneous expansionary effect on output, but also pushes inflation above the target. This induces the central bank to raise the interest rate, which brings down output below equilibrium in the next period. The central bank then gradually lowers the policy rate towards its new higher equilibrium value, where inflation is again stabilised at its target level.

Figure 1: Output



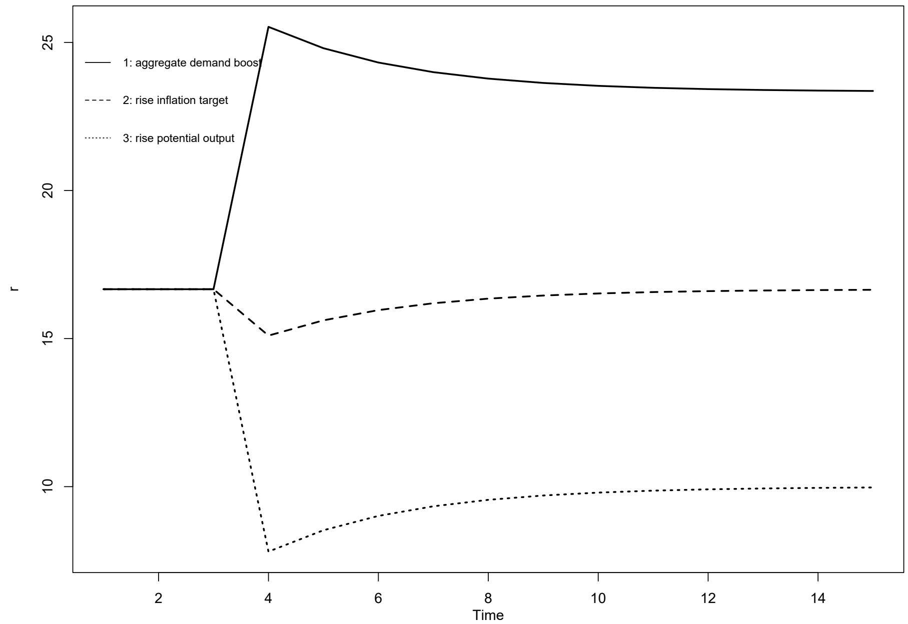
An increase in the central bank's inflation target (scenario 2) gradually raises the inflation rate to a new level. During the adjustment period, the interest rate falls, which temporarily allows for a higher level of output. However, there is no permanent expansionary effect.

Figure 2: Inflation



By contrast, an increase in potential or equilibrium output (scenario 3) allows for a permanently higher level of output and a lower real interest rate.

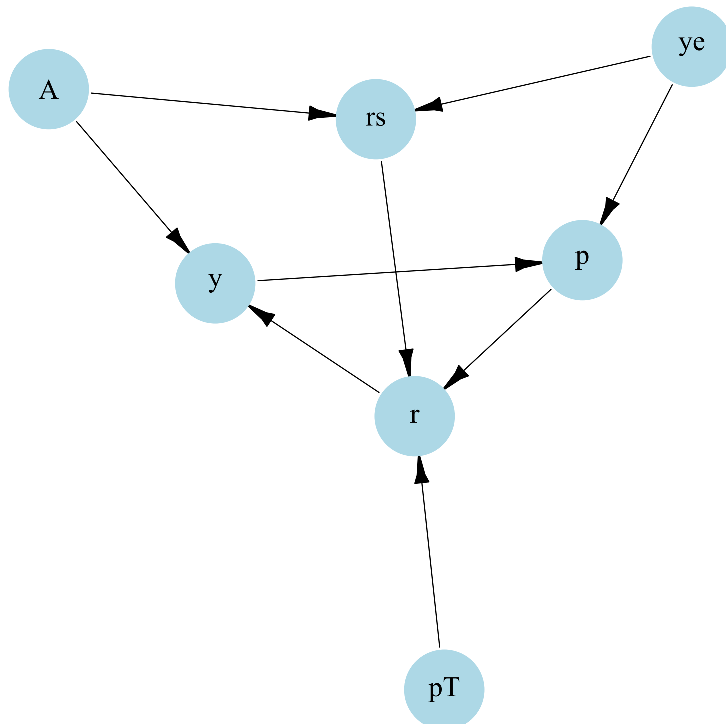
Figure 3: Real interest rate



4 Directed graph

Another perspective on the model's properties is provided by its directed graph. A directed graph consists of a set of nodes that represent the variables of the model. Nodes are connected by directed edges. An edge directed from a node x_1 to node x_2 indicates a causal impact of x_1 on x_2.

Figure 4: Directed graph of 3-equation model



Notes: The directed graph depicts the model's steady state.

In Figure 4, it can be seen that aggregate demand (A), equilibrium output (y_e), and the inflation target (π^T) are the key exogenous variables of the model. All other variables are endogenous and form a closed loop (or cycle) within the system. The upper-right side of the graph represents the supply side, given by the equilibrium level of output and its effect on inflation. The upper-left side captures the demand side and its effect on actual output. The key endogenous variables, output, inflation, and the interest rate form the centre of the graph, where they stand in a triangular relationship to each other. Output drives inflation, which in turn impacts the real interest rate. The latter then feeds back into output. Structural changes in the relationship between demand and supply (e.g. excess demand) also impact the system through their effect on the stabilising interest rate (r_s).

Appendix

A Derivation of core equations

The IS-curve (1) is loosely based on the consumption Euler equation.[4] Suppose there are two periods and the household's utility function is $U = \ln(C_t) + \beta \ln(C_{t+1})$ subject to the intertemporal budget constraint $C_t + \frac{C_{t+1}}{1+r} = Y_t + \frac{Y_{t+1}}{1+r}$. Substituting the constraint into the objective function and differentiating with respect to C_t yields the first-order condition:

$$C_t = \frac{C_{t+1}}{\beta(1+r)}. \tag{5}$$

This consumption Euler equation establishes the negative relationship between the real interest rate and expenditures in (1).

The Phillips curve (2) is derived from wage- and price-setting in imperfect labour markets.[5] Consider the following wage- and price-setting functions:

$$\frac{W}{P^E} = B + \alpha(y_t - y_e) + z_w \tag{6}$$

$$P = (1+\mu)\frac{W}{\lambda}, \tag{7}$$

i.e. the nominal wage W, adjusted for the expected price level, is increasing in the output gap, a factor B capturing unemployment benefits and the disutility of work as well as a vector z_w of wage-push factors. Prices are set based on a constant mark-up (μ) on unit labour cost ($\frac{W}{\lambda}$).

In equilibrium, the real wage is given by: $w_e = B + z_w$. In a dynamic setting, wage setters will raise the expected real wage by $\widehat{\left(\frac{W}{P^E}\right)} = \hat{W}_t - \hat{P}_t^E = \frac{W}{P^E} - w_e = \alpha(y_t - y_e)$. Together with adaptive expectations for prices $\hat{P}_t^E = \hat{P}_{t-1}$, this yields the following equation for wage inflation:

$$\hat{W}_t = \hat{P}_{t-1} + \alpha(y_t - y_e). \tag{8}$$

Transforming equation (7) into growth rates ($\hat{P} = \hat{W}$) and combining it with the wage-

[4]See here for a more detailed derivation.
[5]See Carlin & Soskice (2014, chap. 2) for details.

inflation equation (8) yields the Phillips curve (2).

Finally, to derive the interest rate rule, start from the following central bank loss function:[6]

$$L = (y_t - y_e)^2 + b(\pi_t - \pi^T)^2. \tag{9}$$

Substituting the Phillips curve (2) into the loss function, differentiating with respect to y_t, and simplifying yields the first-order condition:

$$y_t - y_e = -a_2 b(\pi - \pi^T), \tag{10}$$

which can also be regarded as a monetary policy rule. Next, substitute the Phillips curve (2), the IS-curve (1), and the stabilising interest rate (3) into the monetary policy rule and define $a_3 = \frac{1}{a_1(\frac{1}{a_2 b} + a_2)}$, which yields the interest rate rule (4).

B Stability analysis

By definition, in the steady state we have $y^* = y_e$. This implies that $r^* = r_s$. From this, it follows that $\pi^* = \pi^T$.

To analyse the dynamic stability of the model, we rewrite it as a system of first-order difference equations. To this end, substitute (1) into (2), which yields:

$$\pi_t = \pi_{t-1} + a_2(A - a_1 r_{t-1} - y_e). \tag{11}$$

Substitute this equation into (4), which yields:

$$r_t = r_s + a_3[\pi_{t-1} + a_2(A - a_1 r_{t-1} - y_e) - \pi^T]. \tag{12}$$

The Jacobian matrix of the system in (1), (11), and (12) is given by:

$$J = \begin{bmatrix} 0 & 0 & -a_1 \\ 0 & 1 & -a_1 a_2 \\ 0 & a_3 & -a_1 a_2 a_3 \end{bmatrix}. \tag{13}$$

[6]See Carlin & Soskice (2014, chap. 3) for details.

The eigenvalues of the Jacobian can be obtained from the characteristic polynomial $\lambda^3 - Tr(J)\lambda^2 + [Det(J_1) + Det(J_2) + Det(J_3)]\lambda - Det(J) = 0$, where $Tr(J)$ and $Det(J)$ are the trace and determinant, respectively, and $Det(J_i)$ refers to the i_{th} principal minor of the matrix.[7] As there is a row in the Jacobian that only contains zeros, it follows that the matrix is singular and will have a zero determinant. In addition, all principal minors turn out to be zero. The characteristic polynomial thus reduces to $\lambda^2[\lambda - Tr(J)] = 0$. From this, it is immediate that $\lambda_{1,2} = 0$ and $\lambda_3 = Tr(J)$, where $Tr(J) = 1 - a_1 a_2 a_3 = \frac{1}{1+a_2^2 b}$. Stability requires the single real eigenvalue to be smaller than unity (in absolute terms). With $\lambda_3 = \frac{1}{1+a_2^2 b}$, stability thus only requires $a_2 \neq 0$ and $b > 0$, i.e. the output gap needs to impact inflation (otherwise the key channel through which interest rate policy brings inflation back on target is blocked) and the central bank needs to assign a (non-negative) loss to deviations of actual inflation from its target.[8]

C Construction of directed graph

The directed graph can be derived from the model's Jacobian matrix.[9] Let $\mathbf{x}$ be the vector containing the model's variables. The system of equations representing the model can be written as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The Jacobian matrix is then given by $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$.

Next, construct an 'auxiliary' Jacobian matrix $\mathbf{M}$ in which the non-zero elements of the Jacobian are replaced by ones, whereas zero elements remain unchanged, i.e.

$$M_{ij} = \begin{cases} 1 & \text{if } J_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, taking the transpose of the 'auxiliary' Jacobian matrix yields the adjacency matrix ($\mathbf{M^T} = \mathbf{A}$), which is a binary matrix whose elements (A_{ji}) indicate whether there is a directed edge from a node x_j to node x_i. From the adjacency matrix, the directed graph is constructed.

[7]See Gandolfo (2009, chap. 9) for a detailed treatment.

[8]As mentioned in footnote 2, this property of the Carlin-Soskice model is very different from conventional New Keynesian models with rational expectations. In these models, variables such as output and inflation are driven by the 'forward-looking' behaviour of rational agents, i.e. they depend on expectational terms for their current values rather than lagged values. To ensure what is called 'determinancy', these forward-looking variables must adjust fast (or 'jump') to bring the economy back onto a path that is consistent with the optimising equilibrium. This requires the number of jump variables to be matched by an equal number of unstable roots (i.e. being outside the unit circle).

[9]See Fennell et al. (2015) for a neat exposition.

References

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